

Explicit examples of Lipschitz, one-homogeneous solutions of log-singular planar elliptic systems

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Abstract

We give examples of systems of Partial Differential Equations that admit non-trivial, Lipschitz and one-homogeneous solutions in the form $u(R, \theta) = Rg(\theta)$, where (R, θ) are plane polar coordinates and $g : \mathbb{R}^2 \rightarrow \mathbb{R}^m$, $m \geq 2$. The systems are singular in the sense that they arise as the Euler-Lagrange equations of the functionals $I(u) = \int_B W(x, \nabla u(x)) dx$, where $D_F W(x, F)$ behaves like $\frac{1}{|x|}$ as $|x| \rightarrow 0$ and W satisfies an ellipticity condition. Such solutions cannot exist when $|x|D_F W(x, F) \rightarrow 0$ as $|x| \rightarrow 0$, so the condition is optimal. The associated analysis exploits the well-known Fefferman-Stein duality [7]. We also discuss conditions for the uniqueness of these one-homogeneous solutions and demonstrate that they are minimizers of certain variational functionals.

1 Introduction

This paper exhibits explicit Lipschitz one-homogeneous maps $u : \mathbb{R}^2 \rightarrow \mathbb{R}^m$ as solutions to certain systems of nonlinear Partial Differential Equations. In terms of plane polar coordinates, such maps are of the form $u(R, \theta) = Rg(\theta)$, where g is a Lipschitz function taking values in \mathbb{R}^m . The system of nonlinear PDE is

$$\partial_{x_j}(\partial_{F_{ij}} W(x, \nabla u(x))) = 0, \quad i = 1, 2, \quad (1.1)$$

where the summation convention is understood; they are a componentwise form of the Euler-Lagrange equations of the integral functional

$$I(u) = \int_{B_r} W(x, \nabla u(x)) dx.$$

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The integrand $W : B_r \times \mathbb{R}^{m \times 2}$ can be written

$$W(x, F) = f(|F|) + \sum_{1 \leq i < j \leq m} \lambda_{ij} \ln(|x|) \det F^{(i,j)}, \quad (1.2)$$

where all λ_{ij} are constant,

$$F^{(i,j)} = \begin{pmatrix} F_{i1} & F_{i2} \\ F_{j1} & F_{j2} \end{pmatrix}$$

for $1 \leq i < j \leq m$, B_r is the ball with centre 0 and radius r in \mathbb{R}^2 , and $f : [0, \infty) \rightarrow \mathbb{R}$ is a suitably differentiable function. We henceforth write

$$\gamma(F) = f(|F|).$$

The significance of such a result is twofold. Firstly, non-trivial one-homogeneous solutions of the systems (1.1) have long been sought after, and in several cases found, in the context of regularity theory, beginning with the work [18] of Nečas. Here, and in [10],[25], one-homogeneous solutions are in fact minimizers of variational integrals of the form $\tilde{I}(u) = \int_{\Omega} W(\nabla u(x)) dx$ for an appropriate function W , a condition implying stationarity. See also [26] for nonsmooth minimizers which are not one-homogeneous, but which are related to and improve upon the examples in [25]. The domain dimension n in all these examples is at least 3. In contrast, Phillips showed in [19] that one-homogeneous stationary points of functionals like \tilde{I} with $n = 2$ are not possible: the claim in this paper is that they are, provided we allow the integrand W to depend on x as well as $\nabla u(x)$. If we do not insist on one-homogeneity then in two and higher dimensions [17] and [27] have shown that stationary points can in general be nowhere C^1 , which is an extreme form of singularity. These solutions are constructed iteratively and as such are not explicit, an advantage which the mappings we present here do enjoy. The price apparently to be paid for this explicitness is in the x -dependence of the integrands W defined in (1.2) above.

We briefly review one-homogeneous functions and the type of singularity they can produce. By definition, a positively one-homogeneous (henceforth one-homogeneous) function $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies $u(\lambda x) = \lambda u(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda \geq 0$, whence the representation $u(x) = Rg(\theta)$ with $g(\theta) := u(\cos \theta, \sin \theta)$ and $R = |x|$. We also recall that a non-trivial one-homogeneous function is by definition one that is not linear. When it exists, the weak derivative ∇u of a one-homogeneous function u satisfies

$$\nabla u(x) = u(\psi(x)) \otimes \psi(x) + \nabla u(\psi(x)) - (\nabla u(\psi(x))\psi(x)) \otimes \psi(x),$$

where $\psi(x) = \frac{x}{|x|}$. In terms of polar coordinates,

$$\nabla u(R, \theta) = g(\theta) \otimes e_R(\theta) + g'(\theta) \otimes e_\theta(\theta),$$

where $e_R(\theta) = (\cos \theta, \sin \theta)^T$ and $e_\theta(\theta) = (-\sin \theta, \cos \theta)^T$. The gradient clearly depends only on the angular part $g(\theta)$ of u , so that, provided u is not linear, ∇u is discontinuous at the origin. It is in this sense that non-trivial one-homogeneous functions are singular.

Secondly, it confirms that one of the hypotheses in the recent result [2][Theorem 2.1] is sharp. We restate that result here for the reader's benefit.

Theorem 1.1 (Theorem 2.1, [2]). *Let u be a one-homogeneous function belonging to the class $W^{1,2}(B, \mathbb{R}^m)$ and satisfying*

$$\int_{\Omega} A(x, \nabla u(x)) \cdot \nabla \varphi(x) dx = 0 \quad \forall \varphi \in C_c^1(B, \mathbb{R}^m), \quad (1.3)$$

where A satisfies

(H1) $A(x, F)$ is C^1 and uniformly elliptic in the gradient argument F , i.e., for some fixed $\nu > 0$

$$\frac{\partial A_{ij}}{\partial F_{rs}}(x, F) a_i b_j a_r b_s \geq \nu |a|^2 |b|^2$$

for all $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^2$;

(H2) $|x| \partial_{x_i} A(x, F)$ is continuous on $(B \setminus \{0\}) \times \mathbb{R}^{m \times 2}$ for $i = 1, 2$;

(H3) $\lim_{|x| \rightarrow 0} |x| \partial_{x_i} A(x, \nabla u) = 0$ for $i = 1, 2$.

Then u is linear.

The functions W defined above can be chosen so that

$$A(x, F) := D_F W(x, F)$$

solves (1.3) with u a suitable non-trivial one-homogeneous function, and such that it obeys conditions (H1) and (H2) while violating (H3). We infer from this that condition (H3) is necessary. See Lemma 3.1 for details.

It is natural to ask whether there are circumstances under which the one-homogeneous solutions, \bar{u} , say, that we construct are unique. By studying the stationarity condition (1.1) in the planar case, we give a necessary and, under some additional assumptions, sufficient condition for uniqueness. See Propositions 3.2 and 3.5 for details.

In view of the fact that the \bar{u} solves an Euler-Lagrange equation, it is also natural to ask whether these solutions arise as minimizers of appropriate variational problems. It turns out that they do in least two cases: one corresponding to a problem in which functions u competing in the minimization process are constrained to satisfy $\det \nabla u = 1$ a.e. (see Section 4.1), and another corresponding to an unconstrained problem (see Section 4.2) which has some remarkable similarities to a system constructed by Meyers in [13].

The paper is accordingly divided into three parts: Section 2 gives the construction of a general class of one-homogeneous solutions to the PDE problem (1.1); Section 3 considers among other things the question of uniqueness referred to above, and Section 4 is devoted to a variational interpretation of the results of Section 2.

2 One-homogeneous solutions

2.1 Notation

We denote the $m \times n$ real matrices by $\mathbb{R}^{m \times n}$, and unless stated otherwise we sum over repeated indices. Other standard notation includes $\|\cdot\|_{k,p;\Omega}$ for the norm on the Sobolev space $W^{k,p}(\Omega)$, $\|\cdot\|_{p;\Omega}$ for the norm on $L^p(\Omega)$, and $\rightharpoonup, \overset{*}{\rightharpoonup}$ to represent weak and weak* convergence respectively in both of these spaces. Here, Ω is a domain in \mathbb{R}^n . As usual, we denote by $B(a, R)$ the ball in \mathbb{R}^n centred at a with radius R . When the ball has centre zero and radius r we write B_r , and when the radius is 1 we simply write B for B_1 . $\mathcal{H}^1(\Omega)$ represents the Hardy space dual to $BMO(\Omega)$, the space of functions of Bounded Mean Oscillation (see [7, 3]). In keeping with the general literature, we use \mathcal{H}^1 and \mathcal{L}^2 to represent one-dimensional Hausdorff measure and two-dimensional Lebesgue measure respectively. It will be clear from the context whether \mathcal{H}^1 refers to Hardy space or 1-dimensional Hausdorff measure. Unless stated otherwise, the letters a.e. refer to *scd*²—almost everywhere.

The tensor product of two vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ is written $a \otimes b$ and is the $m \times n$ matrix whose (i, j) entry is $a_i b_j$. The inner product of two matrices $F_1, F_2 \in \mathbb{R}^{m \times n}$ is $F_1 \cdot F_2 = \text{tr}(F_1^T F_2)$. This obviously holds for vectors too. Throughout, we use this inner product to define the norm $|F|$ on matrices F via $|F|^2 = F \cdot F$.

In plane polar coordinates (R, θ) the gradient of $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^m$ is

$$\nabla \varphi = \varphi_{,R} \otimes e_R(\theta) + \varphi_{,\tau} \otimes e_\theta(\theta), \quad (2.1)$$

where $e_R(\theta) = (\cos \theta, \sin \theta)^T$, $e_\theta(\theta) = (-\sin \theta, \cos \theta)^T$ and

$$\varphi_{,\tau} = \frac{1}{R} \varphi_{,\theta}.$$

We write $\varphi_{,R}$ for the partial derivative of φ with respect to R , and similarly for $\varphi_{,\theta}$. In this notation the formula

$$\det \nabla \varphi = J \varphi_{,R} \cdot \varphi_{,\tau} \quad (2.2)$$

holds, where J is the 2×2 matrix corresponding to a rotation of $\frac{\pi}{2}$ radians in the plane, i.e.,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Two useful properties of J are that (i) $J^T = -J$, so that in particular $a \cdot Jb = -Ja \cdot b$ for any two $a, b \in \mathbb{R}^2$, and (ii) $\operatorname{cof} A = J^T A J$ for any 2×2 matrix A . For any set E we write χ_E for the characteristic (or indicator) function of E . If E is an \mathcal{L}^2 -measurable set and g a measurable function then the integral average of g over E is

$$\oint_E g(x) dx = \frac{1}{\mathcal{L}^2(E)} \int_E g(x) dx.$$

A similar definition holds with \mathcal{H}^1 in place of \mathcal{L}^2 .

2.2 Construction of general one-homogeneous solutions

Let W be as in (1.2).

Definition 2.1. (*Critical or Stationary point*) We say that u is a critical or stationary point of the functional $I(u) = \int_B W(x, \nabla u(x)) dx$, where W is given by (1.2), if

$$\int_B D\gamma(\nabla u) \cdot \nabla \varphi + \sum_{1 \leq i < j \leq m} \lambda_{ij} \ln R \operatorname{cof} (\nabla u)^{(i,j)} \cdot \nabla \varphi^{(i,j)} dx = 0 \quad (2.3)$$

holds for all $\varphi \in C_c^\infty(B, \mathbb{R}^m)$.

Thus u is a critical point of I if u solves the weak form of the Euler-Lagrange equations (1.1) for the functional I . Note that the weak form makes sense provided both $\ln(R) \nabla u$ and $D\gamma(\nabla u)$ belong to $L^1(B)$, so for now we assume that this is the case.

The aim of the next technical lemma is to rigorously convert (2.3) into the weak form of the equation

$$\frac{f'(c)}{c} \triangle u + \Lambda u_{,\tau} = 0, \quad (2.4)$$

where c is a constant such that $|\nabla u| = c$ a.e., and $\Lambda \in \mathbb{R}^{m \times m}$ is the antisymmetric matrix defined in (2.6) below. When u is one-homogeneous this equation simplifies considerably and can be solved for any choice of the coefficients λ_{ij} : see Proposition 2.1 below.

Lemma 2.1. *Suppose that the function u belongs to $W^{1,1}(B, \mathbb{R}^m)$ and satisfies $|\nabla u| \ln(2 + |\nabla u|) \in L^1(B)$.*

(a) *For $1 \leq i < j \leq m$ let*

$$H_{ij}(F) = \det F^{(i,j)}.$$

Then

$$\int_B \ln R D H_{ij}(\nabla u) \cdot \nabla \varphi \, dx = \int_B (u_{i,\tau} \varphi_j - u_{j,\tau} \varphi_i) \frac{dx}{R} \quad (2.5)$$

for all $\varphi \in C_c^\infty(B, \mathbb{R}^m)$.

(b) *The Euler-Lagrange equation (2.3) becomes*

$$\int_B D\gamma(\nabla u) \cdot \nabla \varphi - \Lambda u_{,\tau} \cdot \frac{\varphi}{R} \, dx = 0,$$

where the constant $m \times m$ matrix Λ is defined by

$$\Lambda_{ij} = \begin{cases} \lambda_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -\lambda_{ji} & \text{if } i > j \end{cases} \quad (2.6)$$

Proof. (a) We begin by establishing that it is sufficient to prove (2.5) for smooth functions u . Extending u by zero outside B , and calling the resulting function u , we may suppose in particular that ∇u has compact support in \mathbb{R}^2 . By Stein's Lemma [23, Section 5.2, p. 23], the assumption $|\nabla u| \ln(2 + |\nabla u|) \in L^1(B)$ then implies that the maximal function $M(|\nabla u|)$ belongs to $L^1(\mathbb{R}^2)$, and hence that ∇u lies in $\mathcal{H}^1(\mathbb{R}^2)$. This enables us to approximate ∇u using smooth gradients as follows. Firstly, by [23, Section 3, Corollary

1 to Theorem 6], $\mathcal{H}^1(\mathbb{R}^2)$ is naturally isomorphic with the Banach space X , where

$$X = \{v \in L^1(\mathbb{R}^2) : R_j v \in L^1(\mathbb{R}^2), j = 1, 2\},$$

and where $R_j v$ denotes the j^{th} Riesz transform of v . Furthermore, X can be normed by

$$\|v\|_X := \|v\|_1 + \|R_1 v\|_1 + \|R_2 v\|_1.$$

Let ρ_ϵ be a standard mollifier sequence, and define $u_\epsilon = \rho_\epsilon * u$. By taking Fourier transforms, it is straightforward to prove that $R_j \nabla u_\epsilon = \rho_\epsilon * R_j \nabla u$ for $j = 1, 2$, so that in particular by standard properties of mollified L^1 functions, $\|R_j \nabla u_\epsilon\|_1 \rightarrow \|R_j \nabla u\|_1$ as $\epsilon \rightarrow 0$ for $j = 1, 2$. It is now evident that ∇u_ϵ converges in the norm of X to ∇u , so that ∇u_ϵ converges strongly to ∇u in $\mathcal{H}^1(\mathbb{R}^2)$.

A short calculation shows that

$$DH_{ij}(\nabla u) \cdot \nabla \varphi = \text{cof } \nabla u^{(i,j)} \cdot \nabla \varphi^{(i,j)},$$

where $u^{(i,j)} = (u_i, u_j)^T$ for $i < j$. Observing that $\ln R$ is a BMO function, and by appealing to the well-known Fefferman-Stein duality $(\mathcal{H}^1)^* = BMO$, [7, Theorem 2, p 145], it follows that the linear functional T_{ij} defined by

$$T_{ij}(\nabla u) = \int_B \ln R \text{cof } \nabla u^{(i,j)} \cdot \nabla \varphi^{(i,j)} dx$$

is continuous on $\mathcal{H}^1(\mathbb{R}^2)$ for each fixed φ . Here we have implicitly used the fact that cof is linear on the 2×2 minors of $\nabla u \in \mathbb{R}^{m \times 2}$. In particular, since $\nabla u_\epsilon \rightarrow \nabla u$ strongly in $\mathcal{H}^1(\mathbb{R}^2)$, the convergence

$$\int_B \ln R \text{cof } \nabla u_\epsilon^{(i,j)} \cdot \nabla \varphi^{(i,j)} dx \rightarrow \int_B \ln R \text{cof } \nabla u^{(i,j)} \cdot \nabla \varphi^{(i,j)} dx$$

as $\epsilon \rightarrow 0$ is immediate.

Now consider the right-hand side of (2.5). Let

$$\tilde{\varphi}(R, \theta) = \varphi(R, \theta) - \frac{1}{2\pi} \int_0^{2\pi} \varphi(R, \alpha) d\alpha$$

and note that provided u is 2π -periodic in θ we have

$$\int_B (u_{i,\tau} \varphi_j - u_{j,\theta} \varphi_i) \frac{dx}{R} = \int_B (u_{i,\tau} \tilde{\varphi}_j - u_{j,\tau} \tilde{\varphi}_i) \frac{dx}{R}. \quad (2.7)$$

Since φ is smooth, it follows in particular that $\frac{\tilde{\varphi}}{R}$ has a removable singularity at the origin, and is otherwise bounded. Writing the right-hand side of (2.7) as

$$\int_B \frac{\tilde{\varphi}^{(i,j)}}{R} \cdot J \nabla u^{(i,j)} e_R dx,$$

it is therefore clear that we can pass to the limit $\epsilon \rightarrow 0$ in

$$\int_B \frac{\tilde{\varphi}^{(i,j)}}{R} \cdot J \nabla u_\epsilon^{(i,j)} e_R dx,$$

and replace $\tilde{\varphi}$ with φ . In summary, it is sufficient to prove (2.5) for smooth functions u .

To that end, in the following we integrate by parts and then use the fact that $\text{cof } \nabla u$ is divergence free whenever u is a smooth planar map.

$$\begin{aligned} \int_B \ln R DH_{ij}(\nabla u) \cdot \nabla \varphi dx &= \int_B \ln R \text{cof } \nabla u^{(i,j)} \cdot \nabla \varphi^{(i,j)} dx \\ &= - \int_B \varphi^{(i,j)} \cdot \text{cof } \nabla u^{(i,j)} \frac{e_R}{R} dx \\ &= \int_B \varphi^{(i,j)} \cdot J u_\tau^{(i,j)} \frac{dx}{R} \\ &= \int_B (u_{i,\tau} \varphi_j - u_{j,\tau} \varphi_i) \frac{dx}{R}, \end{aligned}$$

so proving (2.5).

(b) The Euler-Lagrange equation (2.3) may be written

$$\int_B D\gamma(\nabla u) \cdot \nabla \varphi + \sum_{1 \leq i < j \leq m} \lambda_{ij} \ln R DH_{ij}(\nabla u) \cdot \nabla \varphi dx = 0.$$

Applying (2.5) it follows that

$$\int_B D\gamma(\nabla u) \cdot \nabla \varphi dx + \sum_{1 \leq i < j \leq m} \int_B \lambda_{ij} (u_{i,\tau} \varphi_j - u_{j,\tau} \varphi_i) \frac{dx}{R} = 0.$$

Fix an index k in $\{1, \dots, m\}$ and note that the terms involving φ_k in the second of these integrals are

$$- \int_B \sum_{k < j \leq m} \lambda_{kj} u_{j,\tau} \varphi_k \frac{dx}{R} + \int_B \sum_{1 \leq i < k} \lambda_{ik} u_{i,\tau} \varphi_k \frac{dx}{R},$$

which, in terms of the matrix Λ defined in (2.6), equals $\int_B -\Lambda u_{,\tau} \cdot \frac{\varphi}{R} dx$, as required. \square

Proposition 2.1. *Let the $m \times m$ matrix Λ defined by (2.6) in the statement of Lemma 2.1 be non-zero. Then*

(a) *there exist one-homogeneous solutions $u(R, \theta) = Rg(\theta)$ to*

$$\int_B D\gamma(\nabla u) \cdot \nabla \varphi - \Lambda u_{,\tau} \cdot \frac{\varphi}{R} dx = 0, \quad (2.8)$$

for all $\varphi \in C_c^\infty(B, \mathbb{R}^m)$ such that g obeys the conservation law

$$|g|^2 + |g'|^2 = c^2 \quad (2.9)$$

for some constant c , and such that

- (i) *u is linear if m is an odd integer, or if m is even and Λ has a non-trivial kernel;*
- (ii) *u is non-linear and the set $u(B)$ is homeomorphic to a two dimensional disk in \mathbb{R}^m covered k times, $k \in \mathbb{N} \setminus \{1\}$, provided f is such that*

$$\left(\frac{(k^2 - 1)f'(t)}{kt} \right)^2$$

lies in the (non-zero) spectrum of $-\Lambda^2$ for some $t > 0$.

- (b) *Let f belong to $C^2(\mathbb{R}^+, \mathbb{R})$ and suppose that $f''(t) > 0$ for all $t > 0$, $f'(0+) \geq 0$. Then any one-homogeneous solution $u = Rg$ to (2.8) such that g is of class $C^2(\mathbb{S}^1, \mathbb{R}^m)$ satisfies the conservation law (2.9) for some constant c . In particular, g satisfies*

$$\frac{f'(c)}{c}(g'' + g) + \Lambda g' = 0$$

on $[0, 2\pi]$.

Proof. (a) Let $g(\theta) = x \cos(k\theta) + y \sin(k\theta)$ for fixed vectors $x, y \in \mathbb{R}^m$ and non-zero integer k to be determined as follows. Note first that (2.9) holds with $c^2 = (1 + k^2)|x|^2$ provided $|x|^2 = |y|^2$ or $k^2 = 1$. In either case, (2.9) implies $|\nabla u| = c$, so that (2.8) becomes

$$\int_B \left(\frac{f'(c)}{c}(g'' + g) + \Lambda g' \right) \cdot \frac{\varphi}{R} dx = 0$$

for all $\varphi \in C_c^\infty(B, \mathbb{R}^m)$. Therefore in order to solve (2.8), and hence (2.3), it is sufficient to ensure that g satisfies

$$\frac{f'(c)}{c}(g'' + g) + \Lambda g' = 0. \quad (2.10)$$

We consider two cases, the first of which corresponds to finding a linear solution of (2.8).

Case (a)(i) If $k^2 = 1$ then clearly $g'' + g = 0$, and (2.10) holds only if Λx and Λy both vanish. If m is odd then the skew-symmetry of Λ guarantees the existence of x (and hence of y by taking $y = x$), while if m is even solutions corresponding to $k^2 = 1$ exist only if $\ker \Lambda$ is non-zero.

Case (a)(ii) When $k \neq 1$ equation (2.10) holds only if

$$\begin{aligned}\rho x + \Lambda y &= 0 \\ \rho y - \Lambda x &= 0,\end{aligned}\tag{2.11}$$

where $\rho := \frac{f'(c)(1-k^2)}{kc}$. Necessarily, $\Lambda^2 x = -\rho^2 x$. Note that if this can be solved for $\rho \neq 0$ then defining y by the second of the above equations, namely $y = \frac{1}{\rho} \Lambda x$, yields a solution to the first. Moreover, y so defined automatically satisfies $|y| = |x|$, which is needed in order that (2.9) holds. Therefore it is sufficient to find x such that $\Lambda x = -\rho^2 x$ for some non-zero ρ . To this end, observe that the matrix $-\Lambda^2$ is positive semi-definite and symmetric, and so has a diagonal representation $\text{Diag}(\rho_1^2, \dots, \rho_m^2)$ in terms of an appropriate basis, with ρ_j real for $1 \leq j \leq m$. Therefore there exists a non-zero eigenvalue for Λ^2 provided $\Lambda^2 \neq 0$. But $\Lambda^2 = 0$ only if all its diagonal entries vanish, and since each such entry is the Euclidean norm of a row (equivalently column) of Λ it must be that $\Lambda^2 = 0$ is possible only when $\Lambda = 0$, contradicting our hypothesis. Finally, we suppose that the integer k is such that the condition on f stated in (a)(ii) above holds, so that there is a non-zero eigenvalue ρ_0^2 of Λ^2 such that

$$\frac{(k^2 - 1)f'(t)}{kt} = \rho_0\tag{2.12}$$

for some $t > 0$. Choose x so that $|x| = t(k^2 + 1)^{-\frac{1}{2}}$. Recalling that $c^2 = (1 + k^2)|x|^2$ we see that this choice implies $t = c$, and hence from (2.12)

$$\frac{(k^2 - 1)f'(c)}{kc} = \rho_0.$$

From this it follows that equations (2.11) hold with $\rho = \rho_0$ and x, y as described above.

(b) Let $P(\theta) = |g|^2 + |g'|^2$ and suppose that (2.8) holds. Let

$$z = \frac{f'(\sqrt{P})}{\sqrt{P}}$$

when $P > 0$, and set $z = 0$ when $P = 0$. The assumptions on f made in statement (b) above are then such that z vanishes if and only if P vanishes.

Integrating by parts in (2.8) we obtain

$$\int_B ((zg')' + zg + \Lambda g') \cdot \varphi \, dR \, d\theta = 0$$

for all $\varphi \in C_c^\infty(B, \mathbb{R}^m)$, and the regularity assumptions on g then imply that

$$(zg')' + zg + \Lambda g' = 0 \quad (2.13)$$

at all θ such that $P(\theta) > 0$. Now let $N = \{\theta \in [0, 2\pi] : P(\theta) > 0\}$. We shall show in the following that N is either empty, in which case (2.9) holds trivially, or $N = [0, 2\pi]$.

First note that since Λ is skew, it follows that $g' \cdot \Lambda g' = 0$, and hence from (2.13) that

$$z(g'' \cdot g' + g' \cdot g) + z'|g'|^2 = 0$$

for $\theta \in N$. Furthermore, $z(\theta) > 0$ for $\theta \in N$, so that the latter equation implies in particular that

$$\frac{1}{2}zP' + z'|g'|^2 = 0$$

on N . On using the definition of z given above, we obtain

$$\frac{1}{2}P' \left(z(\sqrt{P}) + \frac{\dot{z}(\sqrt{P})|g'|^2}{\sqrt{P}} \right) = 0 \quad (2.14)$$

on N .

We claim that (2.14) implies that $P' = 0$ on N . Suppose for a contradiction that $P'(\theta) \neq 0$ for some $\theta \in N$. Then, by (2.14),

$$z(\sqrt{P}) + \frac{\dot{z}(\sqrt{P})|g'|^2}{\sqrt{P}} = 0,$$

which holds only if both terms are non-zero, and in particular when $\dot{z}(\sqrt{P}) < 0$. But then

$$\begin{aligned} z(\sqrt{P}) &= |\dot{z}(\sqrt{P})| \frac{|g'|^2}{\sqrt{P}} \\ &\leq |\dot{z}(\sqrt{P})| \sqrt{P} \\ &= -f''(\sqrt{P}) + \frac{f'(\sqrt{P})}{\sqrt{P}}, \end{aligned}$$

which, when the definition of z is recalled, gives $f''(\sqrt{P}) \leq 0$, contradicting the hypothesis on f . It follows that $P' = 0$ on the set N , and since P is continuous it must be that if N is non-empty then it covers all of $[0, 2\pi]$. Clearly (2.9) then holds, which concludes the proof of part (b) of the proposition. \square

3 Properties of and conditions for uniqueness of critical points

In this section we restrict attention to the planar case and consider the functionals

$$G(u) = \int_{B_r} \gamma(\nabla u) + \lambda \ln R \det \nabla u \, dx$$

where $B_r \subset \mathbb{R}^2$ and u belongs to the class

$$\mathcal{A}_p = \{u \in W^{1,p}(B_r; \mathbb{R}^2) : u = \bar{u} \text{ on } \partial B_r, G(u) \in (\infty, +\infty)\}.$$

In addition to the properties of f assumed in previous sections of the paper, we suppose that f obeys a polynomial growth condition of order p , i.e.,

$$c_1|A|^p \leq f(|A|) \leq c_2(1 + |A|^p) \quad \forall A \in \mathbb{R}^2$$

for fixed positive constants c_1, c_2 .

We use Proposition 2.1 to demonstrate that such functionals possess one-homogeneous critical points, henceforth referred to as \bar{u} . That \bar{u} is itself a critical point of G then implies various facts about other, suitably regular critical points of the functional G , should they exist, in the class \mathcal{A}_p for appropriate $p > 1$. For further details see Propositions 3.2 and 3.5 below. In particular, we give a geometric condition which when satisfied implies the uniqueness of the one-homogeneous critical point \bar{u} of G referred to above. The strict monotonicity of the gradient of the integrand γ plays an important role in the calculations and can be inferred from relatively mild assumptions on f : see Lemma 3.1 for details

We begin by finding \bar{u} under the assumptions that f is C^2 on \mathbb{R}^+ and that $f''(t) > 0$ for all $t > 0$. Let us assume that the angular part $g(\theta)$ of \bar{u} is smooth enough to apply part (b) of Proposition 2.1. Then $|\nabla \bar{u}|^2 = c^2$ for some constant c , and

$$\Lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}. \quad (3.1)$$

According to part (a)(i) of Proposition 2.1, linear solutions exist only if Λ has a non-trivial kernel, which it plainly does not. Hence we suppose there is an integer $k > 1$ and $t > 0$ such that

$$\left(\frac{(k^2 - 1)f'(t)}{kt} \right)^2 = \lambda^2$$

as stated in part (a)(ii) of the Proposition. In these circumstances,

$$g(\theta) = x \cos k\theta + y \sin k\theta$$

where $-\Lambda^2 x = \lambda^2 x$ and $y = \frac{1}{\lambda} \Lambda x$. But $-\Lambda^2 = \lambda^2 \mathbf{1}$, so that the choice of $\frac{x}{|x|}$ is free. We may therefore let $x = |x|e_1$ and $y = |x|e_2$, where

$$|x| = t(1 + k^2)^{-\frac{1}{2}},$$

and hence

$$g(\theta) = t(1 + k^2)^{-\frac{1}{2}} e(k\theta)$$

for all θ . The resulting one-homogeneous map is thus proportional to the k -covering map, as is summarised below:

Proposition 3.1. *Let f be C^2 on \mathbb{R}^+ and such that $f''(t) > 0$ for all $t > 0$. Suppose that either f or λ is chosen so that*

$$\frac{(k^2 - 1)f'(t)}{kt} = \lambda$$

holds for some $t > 0$. Then the k -covering map

$$\bar{u}(R, \theta) = t(1 + k^2)^{-\frac{1}{2}} R e(k\theta)$$

is a stationary point of the functional

$$G(u) = \int_{B_r} \gamma(\nabla u) + \lambda \ln R \det \nabla u \, dx$$

for each $r > 0$.

Henceforth we let $\bar{u}(R, \theta) = a R e_R(k\theta)$ be the one-homogeneous stationary point of G found in Proposition 3.1 above, where the constant $a = t(1 + k^2)^{-\frac{1}{2}}$.

Lemma 3.1. *Let f be C^2 on \mathbb{R}^+ and such that $f''(t) > 0$ for all $t > 0$. Let $\gamma(F) = f(|F|)$ be such that $D\gamma(0)$ exists and $\gamma(F) \geq \gamma(0) = 0$ for all $F \in \mathbb{R}^{2 \times 2}$. Then*

- (i) γ is strongly convex, $D\gamma$ is strictly monotone and continuous, and
(ii) if, in addition, $\gamma(F) \geq \nu|F|^2$ for some constant $\nu > 0$ and all F then the functional

$$W(x, F) = \gamma(F) + \lambda \ln R \det F$$

is such that

$$A(x, F) := D_F W(x, F)$$

satisfies hypotheses (H1), (H2) but not (H3) of Theorem 1.1. System (1.3) is solved by the one-homogeneous function \bar{u} defined in Proposition 3.1 above.

Proof. To prove (i), we begin by noting that the hypotheses together with standard results from convex analysis imply that $D\gamma(0) = 0$, $f'(0) = 0$ and that f is strictly convex on $(0, \infty)$. In particular, $f(t) > 0$ and $f'(t) > 0$ for $t > 0$, and f' is continuous on $[0, \infty)$. Now, for any 2×2 matrix Π and any non-zero F ,

$$D^2\gamma(F)[\Pi, \Pi] = f''(|F|)(\Pi \cdot \hat{F})^2 + \frac{f'(|F|)}{|F|} \left(|\Pi|^2 - (\Pi \cdot \hat{F})^2 \right),$$

where $\hat{F} = \frac{F}{|F|}$. Thus $D^2\gamma(F) > 0$ as a quadratic form, and hence every non-zero F is a point of strict convexity of γ , from which we deduce that

$$\gamma(F') - \gamma(F) > D\gamma(F) \cdot (F' - F) \quad \forall F, F' \in \mathbb{R}^{2 \times 2}.$$

(This is standard: see e.g. [20, Theorem 4.5] or [4, Theorem 2.5.2].) It remains to show that the same strict inequality holds for all F' when $F = 0$, which amounts to showing $\gamma(F') > 0$. But this follows easily from the properties of f deduced above. Thus $D\gamma$ is continuous and strictly monotone, and part (i) is proved.

The extra hypothesis supplied in part (ii) ensures that

$$D^2\gamma(F)[\Pi, \Pi] \geq \nu|\Pi|^2$$

for all F . To confirm hypotheses (H1) we must check that

$$D^2W(x, F)[a \otimes b, a \otimes b] \geq \nu|a \otimes b|^2$$

independently of x . But

$$\begin{aligned} D^2W(x, F)[a \otimes b, a \otimes b] &= D^2\gamma(F)[a \otimes b, a \otimes b] + 2\lambda \ln R \det(a \otimes b) \\ &\geq \nu|a \otimes b|^2, \end{aligned}$$

and so (H1) holds. (H2) holds because

$$R\partial_{x_i}D_FW(x, F) = \lambda \operatorname{cof} F$$

is continuous everywhere. Hypothesis (H3), however is violated: if it were to hold we would require

$$\lim_{R \rightarrow 0} R\partial_{x_i}D_FW(x, F) = 0.$$

But by the calculation above this is false whenever $F \neq 0$.

□

The next result will be used in connection with Lemma 3.4.

Lemma 3.2. *Let \bar{u} be as above and assume that u is a critical point of G that is C^1 in a semi-open annulus $\{x \in B_r : r - \delta < |x| \leq r\}$ for some $\delta > 0$. Then*

$$\nabla u(x) = \nabla \bar{u}(x) \quad x \in \partial B_r.$$

Proof. An approximation argument using the regularity assumption on u implies that

$$\int_{B_r} D\gamma(\nabla u) \cdot \nabla \varphi + \lambda \ln R \operatorname{cof} \nabla u \cdot \nabla \varphi \, dx = 0$$

holds in particular when φ merely vanishes at ∂B_r . Similarly,

$$\int_{B_r} D\gamma(\nabla \bar{u}) \cdot \nabla \varphi + \lambda \ln R \operatorname{cof} \nabla \bar{u} \cdot \nabla \varphi \, dx = 0,$$

so that by subtracting the two and letting

$$\Delta = D\gamma(\nabla u) - D\gamma(\nabla \bar{u})$$

the equation

$$\int_{B_r} \Delta \cdot \nabla \varphi + \lambda \ln R \operatorname{cof} (\nabla u - \nabla \bar{u}) \cdot \nabla \varphi \, dx = 0$$

follows. Using (2.8) and (3.1), the term involving $\operatorname{cof} (\nabla u - \nabla \bar{u}) \cdot \nabla \varphi$ can be rewritten as

$$\int_{B_r} \frac{\lambda J(\partial_{,\tau}(u - \bar{u})) \cdot \varphi}{R} \, dx,$$

giving

$$\int_{B_r} \Delta \cdot \nabla \varphi + \frac{\lambda J(\partial_{,\tau}(u - \bar{u})) \cdot \varphi}{R} dx = 0. \quad (3.2)$$

Define the function $\eta(t, \epsilon)$ by

$$\eta(t, \epsilon) = \begin{cases} 0 & \text{if } 0 \leq t \leq r - \epsilon \\ \frac{t - (r - \epsilon)}{\epsilon} & \text{if } r - \epsilon \leq t \leq r, \end{cases}$$

let $\psi = u - \bar{u}$ and take $\varphi = \eta\psi$ in (3.2). The resulting expression involves in particular the term

$$\int_{B_r \setminus B_{r-\epsilon}} \Delta \cdot \frac{\psi \otimes e_R}{\epsilon} dx = - \int_{B_r \setminus B_{r-\epsilon}} \Delta \cdot \left((u_{,R}(r, \theta) - \bar{u}_{,R}(r, \theta)) \otimes e_R + \frac{l(\epsilon)}{\epsilon} \right),$$

where the term $\epsilon^{-1}l(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Using this, (3.2) reads

$$\begin{aligned} & \int_{B_r \setminus B_{r-\epsilon}} \Delta \cdot (\eta(u_{,R} - \bar{u}_{,R}) \otimes e_R - (u_{,R}(r, \theta) - \bar{u}_{,R}(r, \theta)) \otimes e_R) dx + \\ & + \int_{B_r \setminus B_{r-\epsilon}} \frac{\Delta \cdot l(\epsilon)}{\epsilon} dx + \\ & + \int_{B_r \setminus B_{r-\epsilon}} \eta \Delta \cdot (u_{,\theta} - \bar{u}_{,\theta}) \otimes \frac{e_\theta}{R} + \frac{\lambda J(\partial_\tau(u - \bar{u})) \cdot \varphi}{R} dx = 0. \end{aligned} \quad (3.3)$$

Since

$$\frac{1}{\epsilon} \int_{B_r \setminus B_{r-\epsilon}} \eta(x) dx \rightarrow \frac{r}{2} \text{ as } \epsilon \rightarrow 0,$$

it is clear from the assumed regularity of u that

$$\frac{1}{\epsilon} \int_{B_r \setminus B_{r-\epsilon}} \eta \Delta \cdot (u_{,R} - \bar{u}_{,R}) \otimes e_R dx \rightarrow \frac{1}{2} \int_{\partial B_r} \Delta \cdot (u_{,R}(r, \theta) - \bar{u}_{,R}(r, \theta)) \otimes e_R ds.$$

Similarly,

$$\frac{1}{\epsilon} \int_{B_r \setminus B_{r-\epsilon}} \eta \Delta \cdot (u_{,R}(r, \theta) - \bar{u}_{,R}(r, \theta)) \otimes e_R dx \rightarrow \int_{\partial B_r} \Delta \cdot (u_{,R}(r, \theta) - \bar{u}_{,R}(r, \theta)) \otimes e_R ds.$$

Dividing (3.3) by ϵ and letting $\epsilon \rightarrow 0$, it follows that

$$\int_{\partial B_r} \Delta \cdot (u_{,R}(r, \theta) - \bar{u}_{,R}(r, \theta)) \otimes e_R ds = 0. \quad (3.4)$$

Now, in view of $u = \bar{u}$ on ∂B_r , $\nabla u - \nabla \bar{u} = (u_{,R}(r, \theta) - \bar{u}_{,R}(r, \theta)) \otimes e_R$ on ∂B_r , so that (3.4) becomes

$$\int_{\partial B_r} (D\gamma(\nabla u) - D\gamma(\nabla \bar{u})) \cdot (\nabla u - \nabla \bar{u}) ds = 0. \quad (3.5)$$

Since $D\gamma$ is strictly monotone, it follows that (3.5) holds only if $\nabla u = \nabla \bar{u}$ on ∂B_r , concluding the proof. \square

A second and more immediate consequence of the strict monotonicity of $D\gamma$ is contained in the next result. Note that it applies to any two critical points of G .

Proposition 3.2. *Let u_1 and u_2 be critical points of G in \mathcal{A}_p . Then*

$$\int_{B_r} \ln R \det(\nabla u_1 - \nabla u_2) dx \leq 0$$

with equality if and only if $u_1 = u_2$ a.e. in B_r .

Proof. By definition, the critical points u_1 and u_2 satisfy

$$\int_{B_r} D\gamma(\nabla u_j) \cdot \nabla \varphi + \lambda \ln R \operatorname{cof} \nabla u_j \cdot \nabla \varphi dx = 0,$$

$j = 1, 2$, for all smooth test functions φ with compact support in the ball B_r . Subtracting the two equations and letting $w = u_1 - u_2$ yields

$$\int_{B_r} (D\gamma(\nabla u_1) - D\gamma(\nabla u_2)) \cdot \nabla \varphi + \lambda \ln R \operatorname{cof} \nabla w \cdot \nabla \varphi dx = 0$$

for all such φ . By an approximation argument we may take $\varphi = w$ in the above. To be specific, the first term can be approximated by noting that (in view of the assumed p -growth of γ) $D\gamma(\nabla) \in L^{p'}(B_r)$ whenever $\nabla u \in L^p(B_r)$. Here, p' is the Hölder conjugate of p . The second term involving $\ln R$ requires the argument given in the proof of part (a) of Lemma 2.1. The result is that

$$\int_{B_r} (D\gamma(\nabla u_1) - D\gamma(\nabla u_2)) \cdot \nabla w + 2\lambda \ln R \det \nabla w dx = 0.$$

The strict monotonicity of $D\gamma$ together with the fact that $\lambda > 0$ implies

$$\int_{B_r} \ln R \det \nabla w dx \leq 0$$

with equality if and only if

$$(D\gamma(\nabla u_1) - D\gamma(\nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) = 0$$

a.e. in B_r , i.e., if and only if $u_1 = u_2$ a.e. in B_r . \square

The following two results will be of use in connection with our discussion of criteria for uniqueness of critical points of G . The first is essentially an identity; the second is a technical lemma whose function will become apparent in the course of the proof of Proposition 3.5 below.

Lemma 3.3. *Let $u \in \mathcal{A}_p$ and suppose $\bar{u} = aRe_R(k\theta)$ is a one-homogeneous critical point of G . Then*

$$\int_{B_r} \ln R \operatorname{cof} \nabla u \cdot \nabla \bar{u} \, dx = 2\pi k a^2 r^2 \ln r - ak \int_{B_r} u \cdot e_R(k\theta) \, d\theta \, dR. \quad (3.6)$$

Proof. Using the form of gradient given in (2.1), it is straightforward to check that

$$\operatorname{cof} \nabla u \cdot \nabla \bar{u} = Ju_{,R} \cdot \bar{u}_{,\tau} - Ju_{,\tau} \cdot \bar{u}_{,R}.$$

Inserting $\bar{u} = aRe_R(k\theta)$ and integrating gives

$$\begin{aligned} \int_{B_r} \ln R \operatorname{cof} \nabla u \cdot \nabla \bar{u} \, dx &= \int_{B_r} R \ln R Ju_{,R} \cdot ae_R(k\theta) \, dR \, d\theta + \\ &\quad - ak \int_{B_r} \ln R Ju \cdot e_\theta(k\theta) \, dR \, d\theta \\ &= \int_0^{2\pi} [R \ln R Ju \cdot ae_\theta(k\theta)]_{R=0}^{R=r} \, d\theta + \\ &\quad - ak \int_{B_r} (1 + \ln R) Ju \cdot e_\theta(k\theta) \, dR \, d\theta + \\ &\quad + ak \int_{B_r} \ln R Ju \cdot e_\theta(k\theta) \, dR \, d\theta \\ &= k \int_0^{2\pi} r^2 \ln r ae_\theta(k\theta) \cdot ae_\theta(k\theta) \, d\theta + \\ &\quad + ak \int_{B_r} u \cdot Je_\theta(k\theta) \, dR \, d\theta \\ &= 2\pi k a^2 r^2 \ln r - ak \int_{B_r} u \cdot e_R(k\theta) \, dR \, d\theta. \end{aligned}$$

\square

Lemma 3.4. *Let u and \bar{u} be critical points of G in \mathcal{A}_p and suppose that either*

(C1) *$\det \nabla u = \det \nabla \bar{u}$ a.e. in B_r , or*

(C2) *$G(u) = G(\bar{u})$, u is C^1 in a neighbourhood of ∂B_r and γ is homogeneous of degree p , $p \neq 2$.*

Then

$$\int_{B_r} \ln R \det \nabla u \, dx = \int_{B_r} \ln R \det \nabla \bar{u} \, dx, \quad (3.7)$$

and their common value is

$$\int_{B_r} \ln R \det \nabla \bar{u} \, dx = \pi k a^2 (2r^2 \ln r - r^2). \quad (3.8)$$

Proof. If condition (C1) holds then (3.7) is immediate. Therefore assume that (C2) holds. A version of Green's Theorem implies that

$$\int_{B_r} (D\gamma(\nabla u) + \lambda \ln R \operatorname{cof} \nabla u) \cdot \nabla \bar{u} \, dx = r \int_0^{2\pi} (D\gamma(\nabla u) + \lambda \ln r \operatorname{cof} \nabla u) \cdot \bar{u} \otimes e_R(\theta) \, d\theta,$$

and similarly

$$\int_{B_r} (D\gamma(\nabla \bar{u}) + \lambda \ln R \operatorname{cof} \nabla \bar{u}) \cdot \nabla u \, dx = r \int_0^{2\pi} (D\gamma(\nabla \bar{u}) + \lambda \ln r \operatorname{cof} \nabla \bar{u}) \cdot \bar{u} \otimes e_R(\theta) \, d\theta.$$

By Lemma 3.2, we may assume that $\nabla u = \nabla \bar{u}$ on ∂B_r , so that the right-hand sides of the last two equations are equal. In particular, we can then take $u = \bar{u}$ in each (since they hold as identities) and conclude that

$$\int_{B_r} D\gamma(\nabla u) \cdot \nabla u + 2\lambda \ln R \det \nabla u \, dx = \int_{B_r} D\gamma(\nabla \bar{u}) \cdot \nabla \bar{u} + 2\lambda \ln R \det \nabla \bar{u} \, dx.$$

Since γ is assumed to be homogeneous of degree p , it follows that

$$D\gamma(\nabla u) \cdot \nabla u = p\gamma(\nabla u),$$

and similarly for \bar{u} . Hence

$$\int_{B_r} \frac{p}{2} \gamma(\nabla u) + \lambda \ln R \det \nabla u \, dx = \int_{B_r} \frac{p}{2} \gamma(\nabla \bar{u}) + \lambda \ln R \det \nabla \bar{u} \, dx.$$

Now we apply the hypothesis that $G(u) = G(\bar{u})$, which gives

$$\int_{B_r} \gamma(\nabla u) + \lambda \ln R \det \nabla u \, dx = \int_{B_r} \gamma(\nabla \bar{u}) + \lambda \ln R \det \nabla \bar{u} \, dx.$$

Subtracting the two equations and using the assumption $p \neq 2$ implies that

$$\int_{B_r} \gamma(\nabla u) dx = \int_{B_r} \gamma(\nabla \bar{u}) dx,$$

which, since $G(u) = G(\bar{u})$, immediately implies equation (3.7). Finally, it can be checked that the map \bar{u} has constant Jacobian $a^2 k$. Hence equation (3.8). \square

Remark 3.3. *Note that the proof of (3.7) under assumption (C2) also implies that $D\gamma$ is ‘self-adjoint’ on critical points of G in the sense that*

$$\int_{B_r} D\gamma(\nabla u) \cdot \nabla \bar{u} dx = \int_{B_r} D\gamma(\nabla \bar{u}) \cdot \nabla u dx.$$

This holds even when γ is not assumed to be homogeneous of degree p .

Remark 3.4. *Under additional regularity assumptions, [11, 28] have studied the uniqueness problem for critical points of functionals involving a Lagrange multiplier. The singularity associated with $\ln R$ prevents a direct application of their results to the functional G , but perhaps their methods could be adapted to work in this case.*

We now give a uniqueness criterion for critical points of G in \mathcal{A}_p .

Proposition 3.5. *Let u and \bar{u} be critical points of G and suppose that either (C1) or (C2) holds. Then*

$$\int_{B_r} u \cdot e_R(k\theta) dR d\theta \leq \pi a r^2 \tag{3.9}$$

with equality if and only if $u = \bar{u}$ a.e. in B_r .

Proof. The aim is to apply Proposition 3.2. Let $w = u - \bar{u}$ and calculate

$$\begin{aligned}
\int_{B_r} \ln R \det \nabla w \, dx &= \int_{B_r} \ln R \det \nabla u + \ln R \det \nabla \bar{u} \, dx + \\
&\quad - \int_{B_r} \ln R \operatorname{cof} \nabla u \cdot \nabla \bar{u} \, dx \\
&= 2 \int_{B_r} \lambda \ln R \det \nabla \bar{u} \, dx + \\
&\quad - \left(2\pi a^2 r^2 \ln r - ak \int_{B_r} u \cdot e_R(k\theta) \, dR \, d\theta \right) \\
&= \pi k a^2 (2r^2 \ln r - r^2) - 2\pi a^2 r^2 \ln r + \\
&\quad + ak \int_{B_r} u \cdot e_R(k\theta) \, dR \, d\theta \\
&= ak \left(\int_{B_r} u \cdot e_R(k\theta) \, dR \, d\theta - \pi a r^2 \right).
\end{aligned}$$

Here, Lemma 3.4 has been used to replace $\int_{B_r} \ln R \det \nabla u \, dx$ with $\int_{B_r} \ln R \det \nabla \bar{u} \, dx$, and the identity in Lemma 3.3 has been applied to the term involving $\operatorname{cof} \nabla u \cdot \nabla \bar{u}$. In summary,

$$\int_{B_r} \ln R \det \nabla w \, dx = ak \left(\int_{B_r} u \cdot e_R(k\theta) \, dR \, d\theta - \pi a r^2 \right). \quad (3.10)$$

By Proposition 3.2, the right-hand side of (3.10) satisfies

$$ak \left(\int_{B_r} u \cdot e_R(k\theta) \, dR \, d\theta - \pi a r^2 \right) \leq 0$$

with equality if and only if $u = \bar{u}$ a.e. in B_r . Since $ak > 0$, we can now deduce the conclusion of the proposition. \square

Note that the right-hand side of (3.9) is $\int_{B_R} \bar{u} \cdot e_R(k\theta) \, dR \, d\theta$. Therefore one interpretation of (3.9) and Proposition 3.5 is that among all possible critical points of G in \mathcal{A}_p satisfying either (C1) or (C2) it is uniquely \bar{u} which maximizes $\int_{B_r} u \cdot e_R(k\theta) \, dR \, d\theta$.

4 Critical points as minimizers

Functionals of the form

$$G(u) = \int_{B_r} \gamma(\nabla u) + \lambda(x) \det \nabla u \, dx$$

are typically associated with variational problems involving constraints on the Jacobian $\det \nabla u$. In such cases the function $\lambda(x)$ is a Lagrange multiplier, or pressure; it is not necessarily explicitly known *a priori*, though it happens to be in our case, where $\lambda(x)$ is proportional to $\ln|x|$. The fact that \bar{u} constructed in Proposition 3.1 solves the Euler-Lagrange equation

$$\int_{B_r} D\gamma(\nabla \bar{u}) \cdot \nabla \varphi + \lambda \ln R \operatorname{cof} \nabla \bar{u} \cdot \nabla \varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(B_r, \mathbb{R}^2) \quad (4.1)$$

and satisfies $\det \nabla \bar{u} = ak^2$ a.e. in B_r , i.e., is constant almost everywhere, suggests that the stationarity condition (4.1) may well have a variational origin. Could it be that \bar{u} minimizes G among maps in \mathcal{A}_p satisfying $\det \nabla u = ak^2$ almost everywhere? We discuss this question below in Section 4.1.

We point out in Section 4.2 that \bar{u} is in fact the global minimizer of the functional

$$E(u) = \int_{B_r} k^2 |u_{,R}|^2 + |u_{,\tau}|^2 \, dx$$

among *all* functions in \mathcal{A}_2 . There are some remarkable similarities between this and an example given by Meyers in his work [13] on reverse Hölder inequalities in elliptic regularity theory. See the discussion following Proposition 4.2 for details.

4.1 The mapping \bar{u} as a constrained minimizer

Let

$$\mathcal{A}'_p = \{u \in W^{1,p}(B_r, \mathbb{R}^2) : \det \nabla u = 1 \text{ a.e., } u = \bar{u} \text{ on } \partial B_r\}.$$

Here, we will choose the coefficient a in $\bar{u} = aRe_R(k\theta)$ so that $ak^2 = 1$. Hence \bar{u} is also a member of \mathcal{A}'_p . When considered as a functional on \mathcal{A}'_p it is clear that $G(u)$ differs from $\int_{B_r} \gamma(\nabla u)$ only by the constant term $\int_{B_r} \lambda \ln R \, dx$. Therefore for the rest of this section we take

$$G(u) = \int_{B_r} \gamma(\nabla u) \, dx,$$

where γ will be assumed to satisfy the main hypotheses leading to the result of Lemma 3.1, as well as the p -growth hypothesis

$$c_1 |F|^p \leq \gamma(F) \leq c_2 |F|^p + c_3 \quad \forall F \in \mathbb{R}^{2 \times 2}$$

for positive constants c_1, c_2 and c_3 . We assume that $p > 2$ for reasons explained later.

In Theorem 4.1 below we show that \bar{u} minimizes G among all functions u in \mathcal{A}'_p for which the set $u(B_r)$ has the same ‘periodicity’ as \bar{u} . The notion of periodicity is made precise in Lemma 4.1. The technical methods we use are an adaptation of those presented in Sivaloganathan and Spector [21, 22]; we include details of proofs only to keep the paper self-contained and to avoid confusion with the scalings involved. In practice, if the reader is already familiar with [21] then he or she should be able to deduce Theorem 4.1 from Lemma 4.1 and [21, Section 5].

We conjecture that \bar{u} is in fact the global minimizer of G in the full class \mathcal{A}'_p , that is, among functions where no topological control is applied to the image set $u(B_r)$. The techniques of [21] do not seem to work in this case.

In keeping with the notation introduced in [21, 22], let C_R be the circle of radius R and centre zero in \mathbb{R}^2 .

Lemma 4.1. *Let $\phi \in W^{1,p}(B_r; \mathbb{R}^2)$ satisfy $\phi(x) = x$ for all $x \in \partial B_r$ and $\det \nabla \phi = 1$ a.e. in B_r . Define*

$$\phi^{(k)}(R, \theta) = \phi\left(k^{-\frac{1}{2}}R, k\theta\right)$$

for all (R, θ) . Then $\phi^{(k)} \in \mathcal{A}'_p$, and the image set satisfies

$$\phi^{(k)}(B_r) \bigcup_{0 < R \leq r} \bigcup_{0 \leq j \leq k-1} \phi\left(C_{k^{-\frac{1}{2}}R}\right).$$

That is, each set $\phi^{(k)}(C_R)$ is covered k times by the set $\phi\left(C_{k^{-\frac{1}{2}}R}\right)$.

Proof. A calculation shows that

$$\begin{aligned} \det \nabla \phi^{(k)} &= \frac{1}{k^{\frac{1}{2}}R} J\phi_{,R}(k^{-\frac{1}{2}}R, k\theta) \cdot k\phi_{,\theta}(k^{-\frac{1}{2}}R, k\theta) \\ &= J\phi_{,R}(k^{-\frac{1}{2}}R, k\theta) \cdot \frac{\phi_{,\theta}(k^{-\frac{1}{2}}R, k\theta)}{k^{-\frac{1}{2}}R} \\ &= \det \nabla \phi(\rho, \sigma) \end{aligned}$$

where $\rho = k^{-\frac{1}{2}}R$ and $\sigma = k\theta$. Thus $\det \nabla \phi^{(k)} = 1$ a.e. in B_r . Note also that, by definition, $\phi^{(k)}(r, \theta) = \phi(k^{-\frac{1}{2}}r, k\theta)$, so that on applying the hypothesis that ϕ is the identity on ∂B_r , it must be that $\phi^{(k)}(r, \theta) = k^{-\frac{1}{2}}re_R(k\theta)$, i.e., $\phi^{(k)} = \bar{u}$ on ∂B_r . Hence $\phi^{(k)}$ belongs to the class \mathcal{A}'_p .

The last assertion of the lemma follows by noting that $\phi^{(k)}$ is by definition $\frac{2\pi}{k}$ -periodic in the angular variable. Therefore the images $\phi^{(k)}(C_R)$ are formed by overlaying k copies of the sets $\phi\left(C_{k^{-\frac{1}{2}}R}\right)$.

□

Now let

$$\mathcal{D} = \{\phi \in W^{1,p}(B_r; \mathbb{R}^2) : \phi(x) = x \ \forall x \in \partial B_r, \ \det \nabla \phi = 1 \text{ a.e. in } B_r\}$$

and define

$$\mathcal{A}_p'' = \{u \in \mathcal{A}_p' : u = \phi^{(k)}, \ \phi \in \mathcal{D}\}.$$

By Lemma 4.1, $\mathcal{A}_p'' \subset \mathcal{A}_p'$, and note that, by inspection, \bar{u} belongs to \mathcal{A}_p'' .

We show in the next two results that \bar{u} is the global minimizer of G among maps in the class \mathcal{A}_p'' .

Lemma 4.2. *Let $u = \phi^{(k)} \in \mathcal{A}_p''$. Then for a.e. $R \in (0, r)$,*

$$\int_{C_R} \left| \phi^{(k)},_{\tau} \right| d\mathcal{H}^1 \geq 2\pi k^{\frac{1}{2}} R.$$

Proof. By a change of variables, we see that for all $R \in (0, r)$

$$\int_{C_R} \left| \phi^{(k)},_{\tau} \right| d\mathcal{H}^1 = k \int_{C_{k^{-\frac{1}{2}}R}} |\phi,_{\tau}| d\mathcal{H}^1. \quad (4.2)$$

Let $\rho = k^{-\frac{1}{2}}R$. Since $\phi \in W^{1,p}(B_r)$, it is known that $\phi|_{C_\rho} \in W^{1,p}(C_\rho)$ for a.e. $\rho \in (0, r)$. (See [8, Theorem 5.16], for example.) Thus by [12], or see [24, Theorem 1],

$$\int_{C_{k^{-\frac{1}{2}}R}} |\phi,_{\tau}| d\mathcal{H}^1 \geq \mathcal{H}^1(C_\rho) \quad (4.3)$$

for a.e. $\rho \in (0, k^{-\frac{1}{2}}r)$. By [16, Part 2(a), Lemma 3.5], the measure-theoretic boundary of the topological image $\phi^{\text{top}}(B_\rho)$ of B_ρ under ϕ differs from $\phi(B_\rho)$ by at most a set of \mathcal{H}^1 measure zero. We recall that $\phi^{\text{top}}(B_\rho)$ is defined in terms of the Brouwer degree by

$$\phi^{\text{top}}(B_\rho) = \{y \in \mathbb{R}^2 \setminus \phi(C_\rho) : d(\phi, B_\rho, y) \neq 0\}$$

(See e.g. [8] for details of the Brouwer degree.) Thus

$$\mathcal{H}^1(C_\rho) = \mathcal{H}^1(\partial^*(\phi^{\text{top}}(B_\rho))). \quad (4.4)$$

Following [21], the isoperimetric inequality in the plane gives

$$\mathcal{H}^1(\partial^*(\phi^{\text{top}}(B_\rho))) \geq 2\pi^{\frac{1}{2}} \left(\mathcal{L}^2(\phi^{\text{top}}(B_\rho)) \right)^{\frac{1}{2}}. \quad (4.5)$$

If we can show that

$$\mathcal{L}^2(\phi^{\text{top}}(B_\rho)) = \mathcal{L}^2(\phi(B_\rho)) \quad (4.6)$$

then the right-hand side of (4.5) would become $2\pi\rho$. Inserting this into (4.2)-(4.5) would give the desired lower bound $2\pi k^{\frac{1}{2}}R$.

To prove (4.6), we note that because $p > 2$ and because ϕ coincides with a homeomorphism on ∂B_r , by [1, Theorem 1] ϕ is continuous on \bar{B}_r and invertible a.e. in B_r . Then [21, Proposition 5.5(f)] applies to ϕ , so that $d(\phi, B_\rho, y) = 1$ or 0 for all $y \notin \phi(C_\rho)$ for each $\rho \in (0, k^{-\frac{1}{2}}R)$. We also see that the hypotheses of [8, Theorem 5.30] are satisfied, so that in particular

$$\int_{B_\rho} \det \nabla \phi \, dx = \int_{\mathbb{R}^2} d(\phi, B_\rho, y) \, dy.$$

The left-hand side is easily evaluated by setting $\det \nabla \phi = 1$ a.e., giving $\pi\rho^2$. We claim that the right-hand side gives $\mathcal{L}^2(\phi(B_\rho))$. Firstly, we show that $\phi(B_\rho) \setminus \phi^{\text{top}}(B_\rho)$ is a set of \mathcal{L}^2 -measure zero. (Note that $\phi^{\text{top}}(B_\rho) \subset \phi(B_\rho)$ using the definition together with basic properties of the degree.) Let $y \in \phi(B_\rho) \setminus \phi^{\text{top}}(B_\rho)$. We can assume that $y \notin \phi(\partial B_\rho)$ since this set is null (by [8, Theorem 5.32]). Then there is $x_1 \in B_\rho$ such that $\phi(x_1) = y$, and, since y is not in the topological image of ϕ , $d(\phi, B_\rho, y) = 0$. But ϕ agrees with the identity on ∂B_r , so $d(\phi, B_r, y) = 1$ for all $y \in B_r$. Using [8, Chapter 2], for all $y \notin \phi(C_\rho)$

$$\begin{aligned} d(\phi, B_r, y) &= d(\phi, B_r \setminus C_\rho, y) \\ &= d(\phi, B_\rho, y) + d(\phi, B_r \setminus \bar{B}_\rho, y). \end{aligned}$$

It follows that $d(\phi, B_r \setminus \bar{B}_\rho, y) = 1$, and so there must be $x_2 \in B_r \setminus \bar{B}_\rho$ such that $y = \phi(x_2)$. Therefore ϕ is not 1 - 1 at y , and since ϕ is a.e. 1 - 1 it must be that y belongs to a null set. Hence, since $d(\phi, B_\rho, y) = 1$ for all $y \in \phi^{\text{top}}(B_\rho)$, we have

$$d(\phi, B_\rho, y) = \chi_{\phi(B_\rho)}(y) \text{ a.e. } y \in B_r,$$

so that

$$\int_{\mathbb{R}^2} d(\phi, B_\rho, y) \, dy = \mathcal{L}^2(\phi(B_\rho)).$$

□

We now apply Lemma 4.2 to the functional G , following the method of [21, Remark 3.10]

Theorem 4.1. *Let $u \in \mathcal{A}_p''$. Then $G(u) \geq G(\bar{u})$.*

Proof. Let $u = \phi^{(k)}$ for some $\phi \in \mathcal{D}$. Note that since $\det \nabla \phi^{(k)} = 1$ a.e. in B_r , it follows from (2.2) and the Cauchy-Schwarz inequality that

$$|\phi^{(k)},_R| \geq \frac{1}{|\phi^{(k)},_\tau|} \quad (4.7)$$

a.e. in B_r . Following [21], we first show

$$\int_{B_r} |\nabla u| dx \geq \int_{B_r} |\nabla \bar{u}| dx.$$

To that end, let

$$h(s) = (s^2 + s^{-2})^{\frac{1}{2}}$$

for all $s > 0$. Note that h is convex on $(0, \infty)$ and strictly increasing on $[1, \infty)$. By the coarea formula ([6, Theorem 1, Section 3.4.2]), (4.7), and Jensen's inequality,

$$\begin{aligned} \int_{B_r} |\nabla u| dx &= \int_0^r \int_{C_R} \left(|\phi^{(k)},_R|^2 + |\phi^{(k)},_\tau|^2 \right)^{\frac{1}{2}} d\mathcal{H}^1 dR \\ &\geq \int_0^r \int_{C_R} \left(|\phi^{(k)},_\tau|^{-2} + |\phi^{(k)},_\tau|^2 \right)^{\frac{1}{2}} d\mathcal{H}^1 dR \\ &= \int_0^r \int_{C_R} h(|\phi^{(k)},_\tau|) d\mathcal{H}^1 dR \\ &\geq \int_0^r 2\pi R h \left(\int_{C_R} |\phi^{(k)},_\tau| d\mathcal{H}^1 \right) dR. \end{aligned} \quad (4.8)$$

By Lemma 4.2,

$$\int_{C_R} |\phi^{(k)},_\tau| d\mathcal{H}^1 \geq k^{\frac{1}{2}}$$

for a.e. $R \in (0, r)$. Since k is a positive integer, it follows that

$$h \left(\int_{C_R} |\phi^{(k)},_\tau| d\mathcal{H}^1 \right) \geq h(k^{\frac{1}{2}}).$$

Inserting this into (4.8) and using the fact that $\bar{u} = k^{-\frac{1}{2}} R e_R(k\theta)$ satisfies

$$|\nabla \bar{u}| = h(k^{\frac{1}{2}}),$$

we conclude that

$$\int_{B_r} |\nabla u| dx \geq \int_{B_r} |\nabla \bar{u}| dx. \quad (4.9)$$

Finally, since $\gamma(F) = f(|F|)$, where f is convex and increasing, Jensen's inequality combined with (4.9) gives

$$\begin{aligned}
\int_{B_r} \gamma(\nabla u) dx &= \int_{B_r} f(|\nabla u|) dx \\
&\geq \pi r^2 f\left(\int_{B_r} |\nabla u| dx\right) \\
&\geq \pi r^2 f\left(\int_{B_r} |\nabla \bar{u}| dx\right) \\
&= \pi r^2 f(h(k^{\frac{1}{2}})) \\
&= \int_{B_r} \gamma(\nabla \bar{u}) dx.
\end{aligned}$$

□

Remark 4.1. Note that the proofs above also show that \bar{u} is the unique minimizer of G in \mathcal{A}_p'' . This can easily be seen by examining the conditions for equality in inequalities (4.5), (4.7) and in the application (4.8) of Jensen's inequality.

4.2 The mapping \bar{u} as a global minimizer

Define the functional E by

$$E(u) = \int_{B_r} k^2 |u_{,R}|^2 + |u_{,\tau}|^2 dx \quad (4.10)$$

and note that E is well defined on the class \mathcal{A}_2 . Let \bar{u} be as in Section 4.1.

Proposition 4.2. *The map \bar{u} solves the Euler-Lagrange equation*

$$\int_{B_r} u_{,\tau} \cdot \varphi_{,\tau} + k^2 u_{,R} \cdot \varphi_{,R} dx = 0 \quad \forall \varphi \in C_c^\infty(B_r, \mathbb{R}^2) \quad (4.11)$$

associated with the functional E . Consequently, \bar{u} is the unique global minimizer of E in the class \mathcal{A}_2 .

Proof. By definition, the Euler-Lagrange equation for \bar{u} is

$$\partial_\epsilon|_{\epsilon=0} E(\bar{u} + \epsilon \varphi) = 0,$$

where φ is a smooth test function with compact support in B_r . A standard argument yields (4.11). To see that \bar{u} solves it we must have

$$\int_{B_r} k^{\frac{1}{2}} e_\theta(k\theta) \cdot \varphi_{,\tau} + k^{\frac{3}{2}} e_R(k\theta) \cdot \varphi_{,R} R dR d\theta = 0$$

for all such φ . Integrating by parts with respect to θ in the first term and with respect to R in the second, we require

$$\int_{B_r} k^{\frac{3}{2}} e_R(k\theta) \cdot \varphi \, d\theta \, dR + k^{\frac{3}{2}} \int_0^{2\pi} \left([R\varphi \cdot e_R(k\theta)]_{R=0}^{R=1} - \int_0^1 \varphi \cdot e_R(k\theta) \, dR \right) d\theta = 0.$$

Since φ has compact support in B_r and is bounded at 0, the second term vanishes. The remaining terms cancel.

To see that \bar{u} is the unique global minimizer of E in \mathcal{A}_2 note that by (4.11) and a simple approximation,

$$E(v) = E(\bar{u}) + E(v - \bar{u})$$

holds for any $v \in \mathcal{A}_2$. Thus $E(v) \geq E(\bar{u})$ with equality if and only if $E(v - \bar{u}) = 0$, i.e. if and only if $v = \bar{u}$ a.e. in B_r . \square

At first sight it appears that Proposition 4.2 provides us with a non-smooth minimizer of a strongly uniformly elliptic functional E . But when the integrand

$$W(u_{,R}, u_{,\tau}) = k^2 |u_{,R}|^2 + |u_{,\tau}|^2$$

appearing in $E(u)$ is expressed in terms of the derivatives $u_{,x_1}$ and $u_{,x_2}$ we see that

$$W(u_{,R}, u_{,\tau}) = \widetilde{W}(x, u_{,x_1}, u_{,x_2}),$$

where

$$\widetilde{W}(x, u_{,x_1}, u_{,x_2}) = l_1(x) |u_{,x_1}|^2 + l_2(x) u_{,x_1} \cdot u_{,x_2} + l_3(x) |u_{,x_2}|^2$$

and

$$\begin{aligned} l_1(x) &= (k^2 - 1) \cos^2 \theta + 1 \\ l_2(x) &= 2(k^2 - 1) \sin \theta \cos \theta \\ l_3(x) &= (k^2 - 1) \sin^2 \theta + 1. \end{aligned}$$

In particular, the coefficients l_j belong to $L^\infty(B_r) \setminus C^0(B_r)$, so that even though \widetilde{W} is strongly uniformly elliptic in the sense that

$$|\Pi|^2 \leq \frac{\partial^2 \widetilde{W}(x, F)}{\partial F_{ij} \partial F_{st}} \Pi_{ij} \Pi_{st} \leq k^2 |\Pi|^2 \quad \forall F, \Pi \in \mathbb{R}^{2 \times 2}, \quad \forall x \in B_r,$$

classical regularity theory (see e.g. [9, Chapter III, Theorem 3.1]) predicts only that weak solutions of (4.11) should, depending on the setting, belong to

$C^{0,\mu}(B_r)$ for some $0 < \mu < 1$. Crucially, we do not expect higher regularity of weak solutions when the coefficients are not continuous, as they plainly are not in this case. Note that \bar{u} is Lipschitz continuous, and so lies in $C^{0,\mu}(B_r)$ for every $\mu \in (0, 1)$. Thus the example here is consistent with established theory. We also remark that this example is consistent with Morrey's more powerful result [14, Theorem 4.3.1]. (Or see [9, Chapter V, Corollary 3.1].)

Finally, we remark that the system (4.11) and its solution \bar{u} bear a striking resemblance to one constructed by Meyers in [13]. His example, which is discussed below, showed that the improved regularity of elliptic systems with merely L^∞ coefficients is controlled by the 'strength of ellipticity' of the system. We refer the reader to [13] or [9, Chapter V, Theorem 2.5] for further details. Meyers considered the following scalar equation:

$$(au_{,x_1} + bu_{,x_2})_{,x_1} + (bu_{,x_1} + cu_{,x_2})_{,x_2} = 0 \quad (4.12)$$

for $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, and where

$$\begin{aligned} a(x) &= \cos^2 \theta + \mu^2 \sin^2 \theta \\ b(x) &= (1 - \mu^2) \sin \theta \cos \theta \\ c(x) &= \sin^2 \theta + \mu^2 \cos^2 \theta. \end{aligned}$$

Here, $\mu \in (0, 1)$ is a constant. If we let $u = (u_1, u_2)^T$ be such that both u_1 and u_2 solve (4.12) then the weak form of the resulting system is, on B_1 , say,

$$\int_{B_1} \nabla u^T A \cdot \nabla \varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(B_1; \mathbb{R}^2). \quad (4.13)$$

The 2×2 matrix A is given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \mu^2 \end{pmatrix}$$

in the basis $\mathcal{B} = \{e_R \otimes e_R, e_R \otimes e_\theta, e_\theta \otimes e_R, e_\theta \otimes e_\theta\}$. It can be checked that $u_\mu = R^\mu e_R(\theta)$ solves (4.13) for $0 < \mu \leq 1$. Now compare the weak form of (4.11): in the same coordinate system it is

$$\int_{B_r} \nabla u \bar{A} \cdot \nabla \varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(B_1; \mathbb{R}^2),$$

where the 2×2 matrix \bar{A} is given by

$$\bar{A} = \begin{pmatrix} k^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can see that Meyers used the same mechanism to ensure that a, b, c belong to $L^\infty(B_1) \setminus C^0(B_1)$.

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